

Complex Paths and the Hartle Hawking Wave-function for Slow Roll Cosmologies

W. G. Unruh*, Moninder Jheeta†
CIAR Cosmology and Gravity Program
Dept. of Physics
University of B. C.
Vancouver, Canada V6T 1Z1
*email: * unruh@physics.ubc.ca*
† mj227@damtp.cam.ac.uk

Abstract

A large set of complex path solutions for the Hartle Hawking semi-classical wave function are found for an inflationary universe in the “slow roll” regime. The implication of these for the semi-classical evolution of the universe is also studied.

Hartle and Hawking [1] suggested that one could overcome the problem of supplying initial conditions for the universe by supplying a theory of those initial conditions within the context of quantum gravity in the path integral formalism. One of the difficulties with any quantum or classical system is that the theory only predicts the evolution of a system—ie the correlations between the states of the physical attributes of a system at different times. Unfortunately in studying the universe as a whole, there exists no way of determining what the early state of the universe other than by observations made now. There is no way of testing whether or not one’s theory of the early universe is actually correct, only of displaying what initial conditions must have been present at that early time to produce the universe as it is now. Since any physical theory always has a set of initial conditions which can produce any arbitrary final state, one is left with aesthetic judgments as to whether or not the required initial conditions are pleasing or not.

However, within the context of the path integral formulation of quantum Einstein gravity, Hartle and Hawking demonstrated that one could generate a natural theory of a replacement for those initial conditions. Namely, since Einstein’s theory allows changes in the structure of space-time, one could imagine a requirement on the universe that it not have any initial conditions. The universe can simply run out of time into the past, with some unique structure demanded at the time that the universe came into existence.

This suggestion is not tenable within the context of ordinary General Relativity because of the various singularity theorems for Lorentzian space-times. However, for a Euclidean space-time, Einstein’s equations allow singularity free solutions with cross sections representing various possibilities for the universe at some instant in time. Such solutions, while of dubious relevance for the classical theory, can come into their own in the quantum theory, especially the quantum theory formulated in terms of path integrals.

The Hartle Hawking prescription was thus that one could calculate a complex valued function, called the wave-function of the universe, within the path integral formalism. This

function is a function of three geometries— interpreted as the geometry of the universe at some particular time— together with configurations of classical fields on that three geometry. The definition they suggested was that this wave function was to be calculated by means of a path integral in which one integrated over all non-singular four geometries which had only this three geometry as a boundary, together with fields on that 4 geometry such that the value of those fields on the specified three geometry were the given fields. I.e.,

$$\Psi(\mathcal{G}^{(3)}), \Phi(x \in \mathcal{G}^{(3)}) = \int e^{iS(\mathcal{G}^{(4)}, \phi(x \in \mathcal{G}^{(4)}))} \delta\phi \delta\mathcal{G} \quad (1)$$

where the four-geometry $\mathcal{G}^{(4)}$, and the fields ϕ defined on that four-geometry, are non-singular.

While this formulation of the wave-function is certainly appealing intuitively, it suffers from numerous technical problems. Since path integrals even in standard situations are typically taken over highly pathological paths (eg for a free particle over paths which are continuous but not differentiable), the requirement that $\mathcal{G}^{(4)}$ and ϕ be non-singular is difficult to specify exactly. Furthermore, this path integral cannot be evaluated even in very simple cases. Finally, once one has calculated this wave-function, it is unclear how it should be used to actually derive predictions from the theory.

However the purpose of this paper is not to address these difficulties. Rather we will take a standard naive approach to the problem. We will assume that the wave-function is well approximated by the semi-classical approximation, in which one assumes that the extrema of the action, S will dominate the path integral. Furthermore, we will neglect the “determinant” terms, which arise from taking into account the integral over paths which are “near enough” to the classical path that one can make the quadratic approximation to the action of these paths near the semi-classical path. Finally, we will interpret this wave function as giving us a semi-classical Hamilton Jacobi function, which we can then use to predict the semi-classical evolution of the universe.

One’s first reaction is that surely this semi-classical evolution will be simply some classical solution to Einstein’s equation. The difficulty arises in that there exist no classical Lorentzian solutions to Einstein’s equations which are non-singular and have only the given three-geometry plus matter fields as a boundary. The proposal made by HH was therefore that these semi-classical solutions be taken to be a combination of Lorentzian and Euclidean four geometries, matched together at some time. The fiducial example is if we take the simplest situation, namely one in which the only source for the gravitational field is a cosmological constant, and that the geometry be restricted to a maximally symmetric geometry, with three geometries being metric three-spheres. In this case one can join a De-Sitter Lorentzian universe to a four dimensional metric sphere (a Euclidean solution to the Einstein equations with cosmological constant). In order that Einstein equations also be valid across the junction between the solutions it is necessary that the extrinsic curvatures of the three geometries at the join be continuous. However, given some definition of the time, one can characterize the Euclidean solution as having imaginary time, while the Lorentzian has real time. The extrinsic curvature in the Euclidean space is therefore imaginary while it is real in the Lorentzian, and thus matching can be smooth only if the extrinsic curvature is zero [2]. Thus the join between the two solutions must take place at the minimum 3-volume for the De-Sitter space, and the maximum 3-volume for the Euclidean— ie one must match the center of the De-Sitter throat to the equator of the Euclidean sphere.

This four geometry has the feature that at any “time” the three geometry obtained by taking an equal time cross section is a real three geometry, and the vague idea has developed that this is typically what one wants to have happen. However, one begins to quickly realize that, in general (ie in more complicated situations), this is impossible. Essentially, the requirement that one have a real three-geometry at each and every time would require a matching between the Euclidean and Lorentzian regimes such that the momenta corresponding to each of the fields, and to the geometry would all be zero at the junction. If not, the momenta would have to be complex in one or the other of the regimes, and would thus quickly (because of the equations of motion) create complex field strengths and complex metrics. However, as we shall see in very simple cases, there exist no solutions to the classical Euclidean Einstein equations which are both regular and have one three boundary on which all of the momenta of all of the fields and of the geometry are zero.

One is therefore driven to investigating complex three-geometries, and complex four geometries as classical solutions to Einstein’s equations. (We also advise the reader to read the papers by Halliwell and Louko [4] and Hartle and Halliwell [3] which contain a deeper discussion of the motivation behind finding complex solutions. Note also the paper by G. Lyons [5] who qualitatively examined the case where the potential for ϕ is quadratic and arrives at some of the same conclusions we do). In this paper we will investigate the complex solutions for a simple problem, namely one in which most of the degrees of freedom of the geometry and the matter fields are frozen, so that the geometry has a high degree of symmetry (in particular having the symmetry of the metric three-sphere) but in which there exists a matter field given by a scalar field with a non-trivial potential, but with the same spherical symmetry of the metric. The potential for the scalar field will be assumed to be almost flat (ie with small slope) and we will examine the complex solutions as a power series in that slope of the potential. With zero slope this problem of course simply reduces to that originally studied by Hartle and Hawking, since a flat scalar field potential is equivalent to a cosmological constant.

We will find that there are in fact an infinite number of complex semi-classical solutions. Many of them will be analytically related to each other. They will also give an action for the final state of the system which is the same for large classes of the paths. There are however an infinite number of analytically inequivalent paths, many of which lead to different wave-functions.

Furthermore, if we examine the classical solutions implied by these solutions, we find that the classical space-time need not be the same as (in fact because of the complex nature of the extremal solutions, cannot be the same as) any of the extremal paths. We suggest that one interpretation of the Hartle Hawking wave function is that the classical universe has no initial singularity because it bounces. Ie, we suggest that the Hartle Hawking wave function is a way of imposing the requirement on the quantum system that the universe bounce (perhaps due to quantum corrections) before it hits a singularity.

I. THE MODEL

The universe is assumed to have the geometry

$$ds^2 = N(t)^2 dt^2 - a(t)^2 \left(dr^2 + \sin^2(r)(d\theta^2 + \sin^2(\theta)d\varphi^2) \right) \quad (2)$$

with a homogeneous scalar field $\phi(t)$. Since we will be interested in complex extrema to the action, we will assume that all of N the lapse function, a the scale factor, and ϕ the scalar field can be complex. However, we will assume that the real universe has real scale factor a and real scalar field ϕ . Thus the wave function of the universe is a function only of real $a = a_f$ and real $\phi = \phi_f$. Ie, we will calculate $\Psi(a_f, \phi_f)$ via the semi-classical approximation.

The action will be assumed to be the standard Einstein Hamiltonian action,

$$S = \int \left(\pi^{ij} \dot{\gamma}_{ij} + \pi_\phi \dot{\phi} - NH_0 - N_i H^i \right) d^3x dt \quad (3)$$

where

$$H_0 = \left(\frac{1}{\sqrt{\gamma}} \pi^{ij} \pi^{kl} \left(\gamma_{ik} \gamma_{jl} - \frac{1}{2} \gamma_{ij} \gamma_{kl} \right) - \sqrt{\gamma} R + \left(\frac{1}{2} \pi_\phi^2 + \sqrt{\gamma} \left(\frac{1}{2\sqrt{\gamma}} \phi_i \phi_j \gamma^{ij} + V(\phi) \right) \right) \right) \quad (4)$$

$$H_i = \pi_{,j}^{ij} + \Gamma_{jk}^i \pi^{jk} + \pi_\phi \phi_j \gamma^{ij} \quad (5)$$

Because of the symmetry we have assumed for our fields, H_i is automatically equal to zero, and we can write the action in terms of a , ϕ and their conjugate momenta π_a and π_ϕ so that

$$S = \int \pi_a \dot{a} + \pi_\phi \dot{\phi} - NH dt \quad (6)$$

$$H = -\frac{1}{24a} \pi_a^2 - 6a + \left(\frac{1}{2a^3} \pi_\phi^2 + a^3 V(\phi) \right) \quad (7)$$

The Hartle Hawking procedure is often characterized as allowing time to become imaginary. However, we will assume that t is real throughout. The transition to Euclidean space-time will be done by allowing N the lapse function to become purely imaginary. However, in addition to allowing a and ϕ to become complex (as they must) we will also allow N to be not only purely real or purely imaginary but also to be an arbitrary complex function of the real time t . Ie, we will be looking at complex four-metrics and complex fields on a real four dimensional manifold characterized by coordinates t, r, θ, φ .

Despite the simplicity of this action, the extrema are still difficult to find analytically. We will therefore find them by assuming that $V(\phi)$ is almost constant, and can be written as

$$V(\phi) = V_0 + \epsilon \phi \quad (8)$$

We will then assume that the solution is analytic in ϵ and solve our problem as a power series in ϵ . Ie, we assume that $V(\phi)$ is essentially that for inflation, with the solutions we examine being for conditions such that a significant era of inflation takes place.

It will be useful for future work to note that this action has a symmetry. In particular if we allow V_0 to go to $V_0 + \epsilon Q$ and ϕ to go to $\phi - Q$ the action is left invariant.

The extrema are defined by the equations

$$\frac{1}{N} \frac{da}{dt} = -\frac{1}{12a} \pi_a \quad (9)$$

$$\frac{1}{N} \frac{d\phi}{dt} = \frac{1}{a^3} \pi_\phi \quad (10)$$

$$\frac{1}{N} \frac{d\pi_a}{dt} = - \left[\frac{1}{24a^2} \pi_a^2 - 6 - \frac{3}{2a^4} \pi_\phi^2 + 3a^2 V(\phi) \right] \quad (11)$$

$$\frac{1}{N} \frac{d\pi_\phi}{dt} = -a^3 \frac{dV}{d\phi} \quad (12)$$

We also have the fifth equation, an integral of the motion of the above equations and the constraint equation corresponding to the variation of the action with respect to N , of

$$H = \left[-\frac{1}{24a}\pi_a^2 - 6a + \left(\frac{1}{2a^3}\pi_\phi^2 + a^3V(\phi) \right) \right] = 0 \quad (13)$$

Using the equations of motion, this can be written as

$$-6aa'^2 - 6a + \frac{1}{2}a^3\phi'^2 + a^3V(\phi) = 0 \quad (14)$$

where $'$ denotes $\frac{1}{N}\frac{d}{dt}$. We note that N occurs in these equations only in the combination Ndt . We can thus define a new variable τ by

$$\tau = \int N(t)dt \quad (15)$$

where we will take the zero of τ to occur at the “time” when a is zero. Since we are allowing N to be complex, τ will also be a complex function of t . $\tau(t)$ will be a parameterized path through the complex τ plane. Note that these different paths are not simply different coordinates. A coordinate transformation on t cannot create a change in for example the ratio of the real to imaginary part of τ . The paths through τ will be coordinate invariant paths. $'$ will then denote the derivative with respect to τ .

We want the four geometry to be non-singular at $\tau = 0$. Since $\tau = 0$ is defined to be the initial point along the path where $a = 0$, the regularity condition on a will be one on the derivative of a and on the momenta π_a and π_ϕ . Examining the metric, we see that it is regular at $a = 0$ if at that point a goes as τ and τ is purely imaginary so as to create a Euclidean metric. (For example, the curvature, $G_0^0 = 3(a'^2 + 1)/a^2$, is non-singular as a goes to zero only if $a'^2 + 1$ goes to zero at least as fast as a^2 does) Thus, the boundary condition at $\tau = 0$ is that $a' = \frac{da}{d\tau} = \pm i$ at $\tau = 0$. Furthermore, the boundary condition of π_ϕ is that π_ϕ must go to zero, or the equation for π_a cannot drive it to zero sufficiently quickly at $\tau = 0$. We thus have the conditions

$$a(0) = 0 \quad (16)$$

$$a'(0) = \beta i \Rightarrow \pi_a = -12\tau \text{ as } \tau \rightarrow 0 \quad (17)$$

$$\pi_\phi(0) = 0 \quad (18)$$

where $\beta = \pm 1$. We also have the final conditions

$$a(\tau_f) = a_f = \pm |a_f| \quad (19)$$

$$\phi(\tau_f) = \phi_f \quad (20)$$

where however τ_f is a free variable which can be chosen so as to allow these equations to be obeyed. As expected we thus have five boundary conditions on the two variables obeying second order equations, a, ϕ and the third variable τ_f . Both signs for the final radius a_f are valid since it is only a^2 which enters the metric.

The action in terms of the τ variable becomes

$$S = \int_0^{\tau_f} \pi_a a' + \pi_\phi \phi' - H d\tau. \quad (21)$$

We will solve the classical equations of motion in the τ variable and as a power series in the slope of the linear potential, $\epsilon = \frac{dV(\phi)}{d\phi}$. At each order in ϵ , the point τ_f at which $a(\tau_f) = a_f$ will also change. We will define τ_0 as the value of τ_f for $\epsilon = 0$. If q_i is one of our dynamic variables, a or ϕ and if $q_i^{(0)}$ is the zeroth order solution, and

$$q_i = q_i^{(0)} + \sum \frac{\epsilon^n}{n!} \delta^n q_i \quad (22)$$

Then

$$q_i(\tau_f(\epsilon)) = q_{if} \quad (23)$$

gives

$$\delta q_i(\tau_0) = -\delta \tau_f q_i'^{(0)}(\tau_0) \quad (24)$$

$$\delta^2 q_i(\tau_0) = -\delta^2 \tau_f q_i'^{(0)}(\tau_0) - (\delta \tau_f)^2 q_i''^{(0)}(\tau_0) - 2\delta \tau_f \delta q'(\tau_0) \quad (25)$$

We shall use these equations for $q_i = a$ to calculate the expansion of τ_f .

Let us now examine the expansion of the action,

$$S = \int_0^{\tau_f} (\pi_i q_i' - H) d\tau \quad (26)$$

To zeroth order we have

$$S^{(0)} = \int \pi_a^{(0)} a^{(0)'} d\tau \quad (27)$$

We also have, for arbitrary ϵ ,

$$\begin{aligned} \frac{dS}{d\epsilon} &= \frac{d\tau_f}{d\epsilon} [\pi_i(\tau_f(\epsilon), \epsilon) q_i(\tau_f(\epsilon), \epsilon)' - H(\pi_i, q_i, \epsilon)] \\ &\quad + \int_0^{\tau_f} \frac{\partial \pi_i}{\partial \epsilon} q_i' + \pi_i \frac{\partial q_i'}{\partial \epsilon} - \left(\frac{\partial H}{\partial \pi_i} \frac{\partial \pi_i}{\partial \epsilon} + \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial \epsilon} + \frac{\partial H}{\partial \epsilon} \right) d\tau \end{aligned} \quad (28)$$

$$= \frac{d\tau_f}{d\epsilon} [\pi_i(\tau_f(\epsilon), \epsilon) q_i(\tau_f(\epsilon))'] - \left(\pi_i \frac{\partial q_i}{\partial \epsilon} \right) \Big|_0^{\tau_f} - \int_0^{\tau_f} \frac{\partial H}{\partial \epsilon} d\tau \quad (29)$$

where we have used the Hamiltonian equations of motion and the constraint equation $H = 0$ to simplify the equations. But,

$$\frac{d\tau_f}{d\epsilon} q'(\tau_f(\epsilon), \epsilon) + \frac{\partial q_i(\tau_f(\epsilon), \epsilon)}{\partial \epsilon} = \frac{dq_{if}}{d\epsilon} = 0 \quad (30)$$

so we finally have

$$\frac{dS(a_f, \phi_f, \epsilon)}{d\epsilon} = - \int_0^{\tau_f} \frac{\partial H(\pi_i, q_i, \epsilon)}{\partial \epsilon} \quad (31)$$

In our case, we have that

$$\frac{\partial H}{\partial \epsilon} = a^3 \phi \quad (32)$$

But, we also have from the equation of motion for ϕ that

$$a^3 = -\frac{1}{\epsilon} (a^3 \phi')' \quad (33)$$

Thus

$$\frac{dS}{d\epsilon} = \frac{1}{\epsilon} \int_0^{\tau_f} (a^3 \phi')' \phi d\tau \quad (34)$$

$$= \frac{1}{\epsilon} a_f^3 \phi(\tau_f)' \phi_f - \frac{1}{\epsilon} \int_0^{\tau_f} a^3 (\phi')^2 d\tau \quad (35)$$

$$= \frac{1}{\epsilon} \phi_f \int_0^{\tau_f} (a^3 \phi')' d\tau - \frac{1}{\epsilon} \int_0^{\tau_f} a^3 (\phi')^2 d\tau \quad (36)$$

$$= -\phi_f \int_0^{\tau_f} a^3 d\tau - \frac{1}{\epsilon} \int_0^{\tau_f} a^3 (\phi')^2 d\tau \quad (37)$$

$$= -\phi_f \int_0^{\tau_f} \frac{\partial H}{\partial V_0} d\tau - \frac{1}{\epsilon} \int_0^{\tau_f} a^3 (\phi')^2 d\tau \quad (38)$$

However, by exactly the same line or argument which led to eqn 31, we have

$$\frac{\partial S(a_f, \phi_f, V_0, \epsilon)}{\partial V_0} = - \int_0^{\tau_f} \frac{\partial H}{\partial V_0} d\tau \quad (39)$$

Thus we finally have

$$\frac{dS}{d\epsilon} = \phi_f \frac{\partial S}{\partial V_0} - \frac{1}{\epsilon} \int_0^{\tau_f} a^3 (\phi')^2 d\tau \quad (40)$$

We can solve this in part by requiring that V_0 occur in S only in the combination $V_0 + \epsilon \phi_f$.

For our particular problem, the second term on the right is of order ϵ . Thus, the first order term δS is determined by the zero order term, and the first interesting terms is the second order term $\delta^2 S$. Since $\phi' = \epsilon \delta \phi' + O(\epsilon^2)$, the second order term in S is determined by the zeroth order term in a and the first order term in $\delta \phi$.

We will look at the first order solutions for both a and ϕ , namely δa and $\delta \phi$.

II. SOLUTIONS AND ACTIONS

A. Zeroth Order solutions

With our potential, $V(\phi, \epsilon) = V_0 + \epsilon \phi$ we have the equations of motion

$$-6aa'^2 - 6a + \frac{1}{2}a^3\phi'^2 + a^3(V_0 + \epsilon\phi) = 0 \quad (41)$$

$$(a^3\phi')' = -a^3\epsilon \quad (42)$$

The zeroth order solutions give that $a^3\phi'$ a constant. If this constant were non-zero, then ϕ would diverge at $\tau = 0$ at least as $1/\tau^2$ and the solution would not be regular. Thus, to zeroth order, ϕ is a constant, and from the final boundary condition $\phi(\tau) = \phi_f$.

The zeroth order solution for a of

$$a^{(0)'}{}^2 = a^{(0)2}(V_0/6) - 1 \quad (43)$$

is (with the appropriate boundary condition)

$$a^{(0)}(\tau) = \beta \sqrt{\frac{6}{V_0}} \sin(i\sqrt{\frac{V_0}{6}}\tau) \quad (44)$$

We will define $\nu = \sqrt{\frac{V_0}{6}}$, since it will occur time and again, so the solution becomes

$$a^{(0)}(\tau) = \beta \frac{1}{\nu} \sin(i\nu\tau) \quad (45)$$

This solution obeys the boundary condition on a at $\tau = 0$. We also have the boundary condition $a^{(0)}(\tau_0) = a_f$. We readily see that this equation has a number of solutions, since $\sin(i\nu\tau)$ is periodic in imaginary τ with period $2\pi i/\nu$. Since a_f is real, we have, defining

$$\begin{aligned} \tau_R &= \text{Real}(\tau_0) \\ \tau_I &= \text{Imag}(\tau_0) \end{aligned} \quad (46)$$

that τ_R and τ_I obey

$$\beta \cosh(\nu\tau_R) \sin(\nu\tau_I) = \nu a_f \quad (47)$$

$$\cos(\nu\tau_I) = 0 \quad (48)$$

For $a_f > 1/\nu$, this has solutions for τ_I

$$\tau_I = (2n + \text{sign}(a_f)\beta\frac{1}{2})\frac{\pi}{\nu} \quad (49)$$

for all n . In addition, the equation for τ_R

$$\cosh(\nu\tau_R) = \nu|a_f| \quad (50)$$

has both positive or negative solutions for τ_R . In what follows an important variable will be $\cos(i\nu\tau_0)$ whose sign depends on both the sign of a_f and on the sign of τ_R . I will define the sign, γ , (ie, $\gamma = \pm 1$) by

$$\cos(i\nu\tau_0) = \gamma \left(+i|\sqrt{\nu^2 a_f^2 - 1}| \right) \quad (51)$$

for $\nu a_f > 1$. In this case

$$\gamma = \text{sign}(\tau_R)\text{sign}(a_f) \quad (52)$$

For $\nu a_f < 1$, τ_R must be zero, and τ_I , the imaginary part of τ must obey

$$\sin(\nu\tau_I) = -\beta\nu a_f \quad (53)$$

and γ is given by

$$\cos(i\nu\tau_0) = \cos(\nu\tau_I) = \gamma \left(+|\sqrt{1 - \nu^2 a_f^2}| \right). \quad (54)$$

In most of what follows we will assume that we are dealing with $\nu a_f > 1$.

Note that for $\tau_R = 0$ and $\tau_I = m\pi/\nu$, a goes to zero. $m = 0$ corresponds to the initial value $\tau = 0$. For even m , these zeros will be repetitions of the initial state. However, for odd m these zeros will turn out to be singularities in the solutions of the equations at higher orders of ϵ .

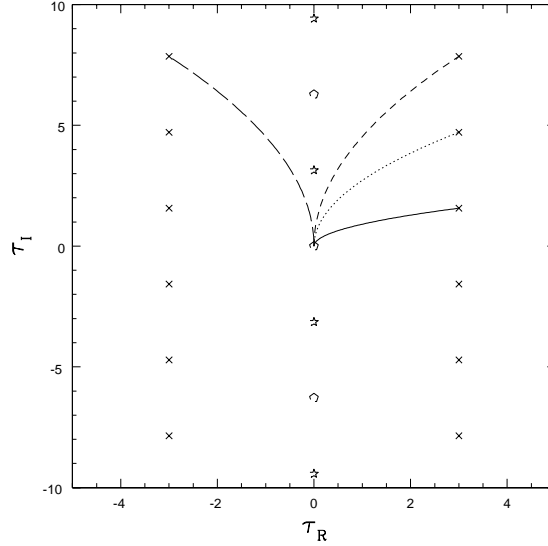


Figure 1

The location of the zeros of $a(\tau)$ and various possible endpoint values for a given a_f of the paths in complex τ space.

Figure 1 shows these poles, the points where a is zero and values of τ_0 which satisfy the $|a^{(0)}(\tau_0)| = |a_f|$. in the complex τ plane. This figure also indicates a few possible paths that the complex function $\tau(t)$ could follow.

B. Zeroth Order action

The zeroth order action is just

$$S(a_f, \phi_f) = \int_0^{\tau_0} \pi_a^{(0)} a^{(0)'} d\tau = - \int_0^{\tau_0} 12a^{(0)} (a^{(0)'})^2 d\tau \quad (55)$$

$$= \frac{24\beta}{V_0} \gamma \left(\frac{V_0 a_f^2}{6} - 1 \right)^{\frac{3}{2}} - i \frac{24\beta}{V_0} \quad (56)$$

There are many different endpoints τ_f which give the same a_f , and many different contours to each of these end points. We note that these contours and end points are “physically” different in that they correspond to different paths through the complex a space. Any one of these solutions can be graphed as a path through the complex a plane. That path is an invariant, and no coordinate transformation can alter that path. They thus represent

physically distinct paths. However, except for the parameters β , γ and $sign(a_f)$ differentiating the different solutions, the actual value of the action as a function of ϕ_f and a_f is independent of which of the paths or of the endpoints are chosen.

In the semi-classical wave function, the real part of the action determines the phase of the wave-function, while the imaginary part determines the amplitude. The two values of γ correspond to taking the complex conjugate of the iS while the two values of β correspond to two different values for the amplitude as well. These two possible values for the amplitude, given by the two possible values of β , have been the subject of immense controversy. We will not take part in this controversy, but will keep β in the remaining equations. We do however note that the issue is not one of determining whether or not the “current” emitted by the $a = 0$ “singularity” is purely outgoing or not. For both signs of β and γ the system obeys the HH condition on the regularity of the wave-function at $a = 0$. Similarly, by taking an appropriate combination of the wave-functions with $\gamma = \pm 1$ one can have a wave function with purely outgoing flux, purely in-going or a combination of the two.

The usual path in τ space, originally chosen in HH, is the path through τ space beginning as a purely imaginary path from 0 to $-i\pi/2\nu$ and then going parallel to the real axis to the endpoint $(cosh^{-1}(\nu a_f)/\nu, -i\pi/2\nu)$. $a^{(0)}(\tau)$ is real along the whole path. Along any other path, a has an imaginary part. We note that this path also corresponds to only one of the possible end values τ_0 . Other end points correspond to trajectories in the complex a plane which wrapping around $a = 0$ more. In figure 2 we plot the trajectories in the complex a plane which correspond to the τ trajectories of figure 1.

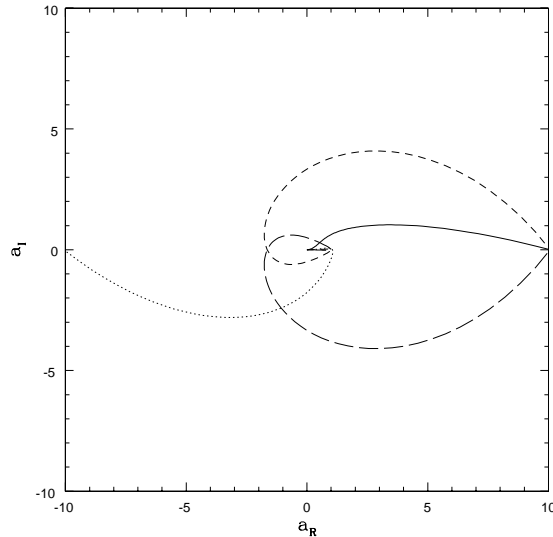


Figure 2

The paths in complex scale factor (a) space of solutions for the various complex time (τ) paths of figure 1.

The function, $S_0(a_f, \phi_f)$ is a Hamilton-Jacobi function for the $\epsilon = 0$ system, in that it obeys the equation

$$-\frac{1}{24a} \left(\frac{\partial S_0}{\partial a} \right)^2 - 6a + \left(\frac{1}{2a^2} \frac{\partial S_0}{\partial \phi} \right)^2 + a^3 V_0 = 0. \quad (57)$$

which is the Hamilton-Jacobi form of the constraint equation for $\epsilon = 0$.

The first order action is just given by

$$\delta S(a_f, \phi_f, V_0) = \frac{\partial S_0(a_f, \phi_f, V_0)}{\partial V_0} \quad (58)$$

This again solves the Hamilton-Jacobi function to first order

$$\frac{1}{12a} \frac{\partial S_0(a, \phi)}{\partial a} \frac{\partial \delta S(a, \phi)}{\partial a} + a^3 \phi = 0 \quad (59)$$

C. First Order Solutions

The first order equations for a and ϕ are required to find the second order action. Actually, only the first order solution in ϕ is needed, but we will also solve the first order equations for a for future reference. The equations are

$$\begin{aligned} -12a^{(0)'} \delta a' + 2V_0 a^{(0)} \delta a + a^{(0)2} \phi_f &= 0 \\ (a^{(0)3} \delta \phi')' &= -a^{(0)3} \end{aligned} \quad (60)$$

with the boundary conditions that all of the variations are zero at $\tau = 0$. Because $\phi^{(0)}$ is a constant, we also have that $\delta \phi(\tau_f) = 0$.

The solution for a is

$$\delta a = \beta \frac{\phi_f}{2V_0} \left(i\tau \cos(i\nu\tau) - \frac{1}{\nu} \sin(i\nu\tau) \right) \quad (62)$$

while for $\delta \phi$ we get first that

$$\begin{aligned} \delta \phi' &= -\frac{1}{a^{(0)3}} \int_0^\tau a^{(0)3} d\tau \\ &= \frac{\frac{i}{\nu} \left(\frac{1}{3} \cos^3(i\nu\tau) - \cos(i\nu\tau) + \frac{2}{3} \right)}{\sin^3(i\nu\tau)} \end{aligned} \quad (63)$$

This expression has poles of order 3 at the points $\tau = \frac{i}{\nu}(2r+1)\pi$

Note that at the end point, $\tau = \tau_0$, the expression for δa not only depends on the value of a_f , but also on which particular end point τ_0 is chosen. This implies also that $\delta \tau = \frac{d\tau_f}{d\epsilon} = -\delta a(\tau_0)/a^{(0)}(\tau_0)'$ will depend on which end point is chosen. Although neither of these enter into the expression for the second order action, we do expect them to have an effect on the third order action, giving an end-point dependence to the action to higher orders.

Because $\delta \phi(\tau_0) = 0$, we have

$$\delta \phi(\tau) = \int_\tau^{\tau_0} \delta \phi' d\tilde{\tau} \quad (65)$$

$$\delta \phi' = -\frac{1}{a^{(0)3}} \int_0^\tau a^{(0)3} d\tilde{\tau} \quad (66)$$

Because of the $1/a^3$ term in the integrand for $\delta\phi$, its value will depend, among other things, on the way in which the path in τ space wraps around the poles where $a = 0$. We get

$$\delta\phi(\tau) = \frac{2}{V_0} \left[\frac{1}{1 + \cos(i\nu\tau)} + \ln \left(\frac{1 + \cos(i\nu\tau)}{1 + \cos(i\nu\tau_0)} \right) - \frac{1}{1 + \cos(i\nu\tau_0)} \right] \quad (67)$$

where the branches of the \ln function around the singularities where $1 + \cos(i\nu\tau) = 0$ are chosen so as to make the expression $\delta\phi(\tau(t))$ continuous along the path $\tau(t)$.

D. Second Order Action

Given the first order solutions, we can calculate the action to second order in ϵ .

$$\delta^2 S = \phi_f^2 \frac{d^2}{dV_0^2} S_0(a_f, \phi_f, V_0) - \int_0^{\tau_0} a^{(0)3} (\delta\phi')^2 d\tau \quad (68)$$

We have

$$- \int_0^{\tau_0} a^{(0)3} (\delta\phi')^2 d\tau = \beta \int_0^{\tau_0} + \frac{1}{\nu^5 \sin^3(i\nu\tau)} \left(\frac{1}{3} \cos^3(i\nu\tau) - \cos(i\nu\tau) + \frac{2}{3} \right)^2 d\tau \quad (69)$$

$$= + \frac{i\beta}{9\nu^6} \int_1^{\cos(i\nu\tau_0)} \left(\frac{(1-z)(z+2)}{1+z} \right)^2 dz \quad (70)$$

with $z = \cos(i\nu\tau)$ which has poles of order 2 in the integrand at $z = -1$ or $\tau = i(2m+1)\pi/\nu$. Each time the integrand circles one of these poles, we accumulate a residue, and it is clear that the residue is the same at each of the poles in τ space, since it is the same pole $z = -1$ in z space. Thus we have

$$- \int_0^{\tau_0} a^{(0)3} (\delta\phi')^2 d\tau = +\beta \frac{i}{9\nu^6} \int_{\Gamma_0} \left(\frac{(1-z)(z+2)}{1+z} \right)^2 dz + 2\pi\beta k \frac{96}{V_0^3} \quad (71)$$

where Γ_0 is some fiducial path from the point $\tau = 0$ to $\tau = \tau_0$, and k is the total net number of times that the actual path wraps around the various poles. (For an encircling of each of the poles in the counter clockwise direction, one gets the same contribution no matter which of the poles is chosen.) Note that the contribution from each of the poles to the action is real. They will therefore contribute only to the phase, and not the amplitude, of the wave-function.

Assuming that $\nu a_f > 1$, we can take the fiducial path as a straight line connecting $\tau = 0$ to $\tau = \tau_f$. We finally get

$$\delta^2 S = \beta \left[\phi_f^2 \frac{\partial^2 S_0(a_f, \phi_f, V_0)}{\partial V_0^2} - \pi k \frac{192}{V_0^3} \right] \quad (72)$$

$$+ \frac{8i}{V_0^3} \left[-\frac{12}{Z+1} + 12 \ln \left(\frac{Z+1}{2} \right) + 17 - 12Z + Z^3 \right] \quad (73)$$

where $Z = \cos(i\nu\tau_0) = \gamma i \sqrt{\frac{V_0 a_f^2}{6} - 1}$.

We note that once again, the action does not depend on the endpoint of the integral—ie on which of the end values of τ_0 the integrand finishes on.

We find that $\delta^2 S$ also obeys the second order Hamilton Jacobi function for the Einstein action. However it is important to note that to second order, S is a complex function of a_f, ϕ_f , with the imaginary part an actual function of these variables, instead of being a constant (as it is to zeroth order). Since the Hamilton Jacobi equation is a non-linear equation to second order in ϵ , the real part of S will not be a Hamilton Jacobi function for the Einstein action. This will be important in the next section where we look at the classical paths which the wave-functions for the universe imply.

III. CLASSICAL EVOLUTION

The impression is often conveyed that the semi-classical path integral implies that the history that the universe follows is just the classical solution which is used to dominate the path integral. It is clear that this cannot be the correct answer. We have seen that the wave-function of the universe has contributions not just from real solutions to Einstein's equations, but also from complex solutions. These complex solutions clearly cannot represent a history of the universe, since physical things like the radius of the universe, the value of the scalar field, and their velocities in (proper) time are real quantities, not complex quantities. The semi-classical approximation to the path integral is a mathematical trick for evaluating the wave function, and does not in itself represent anything “real”, and in particular does not represent a real history. However, once one has a wave function, one can ask the quantum mechanical question as to what types of history of the universe that wave function will represent. This is a very deep problem which does not have any satisfactory solution as yet. However, this paper is not the place for a detailed examination of the various possibilities. Instead we will take a naive approach.

In ordinary quantum mechanics, the momentum operator conjugate to the position is just the imaginary derivative of the wave function. Although this operator really only makes sense if integrated over the configuration space, we will assume that some sense can be made of the momentum density, $\psi(q)^* i \frac{\partial}{\partial q} \psi(x)$. However this quantity is not real, while a physical momentum should be real. Furthermore, the amplitude squared of the wave function, $|\psi(q)|^2$ is the probability of finding the particle at the position q . The derivative of the amplitude of the wave function is thus not really something associated with the momentum of the particle. We can therefore take as the definition of the momentum at the point x , the quantity $\tilde{\psi}(q) i \frac{\partial}{\partial q} \tilde{\psi}(q)$, where $\tilde{\psi}(q) = \psi(q)/|\psi(q)|$. Thus the momentum is just the derivative of the phase of the wave function. If the wave function is of the form e^{iS} , then the momentum will be just the derivative of the **real** part of the action S . This argument is clearly highly suspect, and is at best only a crude approximation. However, it is the approximation which we shall use. Thus, the Real part of the action S will be used as a Hamilton Jacobi function for the evolution of the system as an approximation to the classical evolution implied by the quantum wave function. Again, we emphasize most strongly that this is at best a very crude attempt at deriving a classical evolution from the quantum system. In particular, in cases of significant quantum interference, this approximation will break down entirely. (for example, for a purely real wave-function, this gives the result that the momentum is everywhere zero).

The lowest order action is complex, but the imaginary terms do not depend on either a_f or ϕ_f . In the first order action, the imaginary part does depend on ϕ_f , but since the zeroth order action is entirely independent of ϕ_f , and the Hamilton Jacobi equation has $\frac{\partial S}{\partial \phi_f}$ to the second order, this derivative of the imaginary part of the action will not contribute to the first order HJ equation. Thus, the real part of the first order action will again obey the HJ equation to first order. However, to second order, this is no longer true. The imaginary part of the first order action depends on ϕ_f and thus comes in in quadratic order in the second order HJ equation. Ie, the real part of the second order action will not be a solution of the HJ equation, even though the full complex second order action is. Thus, the histories obtained by using only the real part of the action will not be solutions of the Einstein equations. Einstein's equations will have "quantum corrections" arising from the HH specification for the wave-function of the universe.

In particular, if we use the real part of our second order solution, it obeys the HJ equation to second order in epsilon of

$$-\frac{1}{12a} \left[\left(\frac{\partial \delta S_R}{\partial a} \right)^2 + 2 \frac{\partial S_0}{\partial a} \frac{\partial \delta^2 S_R}{\partial a} \right] + \frac{1}{a^3} \left(\frac{\partial \delta S_R}{\partial \phi} \right)^2 \quad (74)$$

$$= -\frac{1}{12a} \left(\frac{\partial \delta S_I}{\partial a} \right)^2 + \frac{1}{a^3} \left(\frac{\partial \delta S_I}{\partial \phi} \right)^2 \quad (75)$$

But in order for S_R to be a solution of the HJ equation for Einstein's equations to second order in ϵ , the right hand side would have to be zero. Instead it is $\frac{576}{V_0^4 a^3}$. We could subtract a term from the Einstein Hamiltonian constraint to find a new Hamiltonian which the real part would satisfy. We note that this term represents a potential which diverges as a goes to zero, a potential which will help prevent the universe from collapsing.

$$H_{mod} = -\frac{\pi_a^2}{24a} - 6a + \frac{\pi_\phi^2}{2a^3} + a^3 V(\phi) - 288 \frac{\epsilon^2}{V_0^4 a^3} \quad (76)$$

Note that the "quantum" correction falls off rapidly with a , which implies that once the universe is large, the impact of the imaginary parts of the action on the evolution of the universe become negligible.

Let us examine the classical solutions obtained by using S_R as a HJ function for the solution. We are interested in the motion for $\nu|a_f| > 1$, Furthermore, we expect V_0 to enter into the complete action only in the combination $V = V(\phi_f) = V_0 + \epsilon\phi_f$. Thus, we define

$$y = \left| \sqrt{\frac{Va_f^2}{6}} - 1 \right| \quad (77)$$

and have

$$S_R \approx +\beta\gamma \frac{24}{V} y^3 + \epsilon^2 \beta \frac{4}{V^3} \left(-12 \frac{\gamma y}{1+y^2} + 6i \ln \frac{1+i\gamma y}{1-i\gamma y} + 12\gamma y + \gamma y^3 - 24\pi k \right) \quad (78)$$

(Note that the term, $6i \ln \frac{1+i\gamma y}{1-i\gamma y}$ could also be written as $-12\gamma \arctan(y)$ which makes it clear that it is real). The equations relating the derivatives with respect to time of the dynamic variables and the HJ function are

$$\pi_a = -12a\dot{a} = \frac{\partial S_R}{\partial a} \quad (79)$$

$$\pi_\phi = a^3\dot{\phi} = \frac{\partial S_R}{\partial \phi} \quad (80)$$

where we have chosen the time to be the proper time in the space-time (ie, $N=1$).

For large a , we have

$$S_R \approx \beta\gamma|a|^3 4\sqrt{\frac{V(\phi)}{6}} \left[1 + \frac{1}{6V^2}\epsilon^2\right] \quad (81)$$

and the evolution of $\phi \approx 2\epsilon\sqrt{\frac{6}{V_0}}t$, while a grows exponentially. These are just the “slow roll” results for the dynamics in an inflationary system. ϵ makes only a small change in the results.

What is more interesting is the behaviour of a and ϕ near the minimum, when $V(\phi)a^2/6$ is of order unity. We first note that we can rewrite the expression for S_R as

$$S_R \approx \beta\gamma\frac{24}{V} \left((1 + \epsilon^2\frac{17}{6V^2})(y^2 - \epsilon^2\frac{4}{3V^2})^{\frac{3}{2}} \right) + \epsilon^2 O(y^5) + O(\epsilon^3) \quad (82)$$

Ie, to second order in ϵ and to fourth order in y this expression is identical to the previous expression for S_R , as can be readily verified by expanding this expression to second order in ϵ . Since

$$y^2 - \epsilon^2\frac{4}{3V^2} = \frac{V_0}{6}a^2 - 1 - \epsilon^2\frac{4}{3V^2} + \epsilon\frac{1}{6}\phi a^2 \approx \frac{V_0}{6}(a - a_T)(2a_T) + \epsilon\frac{1}{6}\phi a_T^2 + O((a - a_T)^2) \quad (83)$$

where

$$a_T = \sqrt{\frac{6}{V_0}}(1 + \epsilon^2\frac{2}{3V_0^2}) \quad (84)$$

we can write the action as

$$S_R \approx \beta\gamma\frac{24}{V} \left((1 + \frac{17}{6V^2}\epsilon^2) \left(\frac{V_0 a_T}{3}(|a| - a_T) + \epsilon\frac{a_T^2}{6}\phi \right)^{\frac{3}{2}} \right) \quad (85)$$

Defining $\xi = |a| - a_T$, we have

$$\pi_a = -12a\dot{a} \approx -12a_T\dot{\xi} \approx \beta\gamma\frac{24}{V_0}(1 + \epsilon^2\frac{17}{6V_0^2})\frac{3}{2}\frac{V_0 a_T}{3}(\frac{V_0 a_T}{3}\xi + \epsilon\frac{a_T^2}{6}\phi)^{\frac{1}{2}} \quad (86)$$

$$\approx \beta\gamma 12a_T(1 + \epsilon^2\frac{17}{6V_0^2})(\frac{V_0 a_T}{3}\xi + \epsilon\frac{a_T^2}{6}\phi)^{\frac{1}{2}} \quad (87)$$

$$\pi_\phi = a_T^3\dot{\phi} \approx \epsilon\beta\gamma\frac{24}{V_0}\frac{3}{2}\frac{a_T^2}{6}(\frac{V_0 a_T}{3}\xi + \epsilon\frac{a_T^2}{6}\phi)^{\frac{1}{2}} \quad (88)$$

$$\approx \beta\gamma\epsilon\frac{6a_T^2}{V_0}(\frac{V_0 a_T}{3}\xi + \epsilon\frac{a_T^2}{6}\phi)^{\frac{1}{2}} \quad (89)$$

where we have only kept terms to lowest order in ξ and ϕ . Solving these equations we find that

$$\left(\frac{V_0 a_T}{3}\xi + \epsilon \frac{a_T^2}{6}\phi\right)^{\frac{1}{2}} = -\beta\gamma \frac{V_0 a_T}{6} \left(1 + \epsilon^2 \frac{35}{6V_0^2}\right)t \quad (90)$$

Since $\left(\frac{V_0 a_T}{3}\xi + \epsilon \frac{a_T^2}{6}\phi\right)^{\frac{1}{2}}$ was taken as being positive, we have that $\text{sign}(t) = -\beta\gamma$. Ie, the choice of β and γ determines whether the solution starts or ends and that minimal value of $a = a_T$. We also note that ϵ only causes small (or order ϵ^2) changes in the solution.

We finally have

$$|a| - a_T \approx \left(\frac{1}{2}a_T + O(\epsilon^2)\right)t^2 \quad (91)$$

$$\phi \approx -\epsilon \frac{1}{2}t^2 \quad (92)$$

Note that the main effect of the HH boundary conditions has been to demand that ϕ is a maximum in time at the time when a is a minimum. One might have expected $\dot{\phi}$ to be non-zero at this minimum, but it is not. This means that for these states, the value of ϕ cannot be used a clock near the turn around point, since the clock changes sign at point, although in principle π_ϕ could be.

Of course, this turning point is also where one would expect to find maximal quantum interference between the various possible semi-classical paths, and so this non-analytic behaviour of the classical solution is suspect. However we note that the semi-classical wavefunction, if we take an equal linear combination of terms with $\gamma = \pm 1$, can be constantly interpreted not as a wave function which emerges out of the $a = 0$ “singularity” (that would be consistent with taking only the solution with $\gamma = -\beta$), but rather with the boundary condition imposing the condition that the classical universe suffers a bounce. Ie, the HH conditions could be interpreted as enforcing a bouncing universe, rather than as describing the creation of the universe out of nothing.

We note that we have neglected the contribution of the fluctuations to the path integral in all of the above analysis. One would expect that these too would be analytic in the path through τ space which one chooses. However, these fluctuation terms could depend on which of the “end points” the path in τ space ended at. In fact, because the first order solution $\delta a(\tau)$ is not periodic in imaginary τ , one would expect the higher order actions to also depend on which of these end-points was chosen in defining the semi-classical action. Both the fluctuations and the end-point dependence could substantially change the behavior of the wave function, especially near the “turning point”.

IV. CONCLUSION

We have solved the equations for the evolution of the universe in an inflationary approximation, where the slope of the potential driving the scalar field can be taken to be small. There exist an infinite number of complex solutions which can be said to dominate the path integral. Although some of these paths are analytically related to each other, others are not. The analytically related paths, which all have the same action, could, as others have done,

be regarded as equivalent in some sense. Since the manifold is real they are not related by a diffeomorphism, and the question as to whether or not they should be summed over or only one representative of the class should be used in defining the wave-function is to us still an open question. However, there exist singular points for the solutions in the complex τ plane, which correspond to a going to zero. The path in the complex plane can wrap around these points, and the action depends on how the paths wrap around these singularities. To the order of approximation we examine, these singularities contribute to the real part of the path integral to second order in the slope of the potential. There exist an infinite number of these inequivalent analytic paths. Furthermore, there are also an infinite variety of analytically inequivalent paths which depend on the choice of the end point of the integration in the complex τ plane. The action, to second order, depends only on which of two classes the ends points fall into (as designated by the parameter γ above), although there are indications that to higher order the action will also depend on exactly which of these end points is chosen.

If one assumes that the classical path that a universe would follow is best approximated by using the real part of the semi-classical complex action as the Hamilton Jacobi function, then one interpretation of the Hartle-Hawking wave function is not that the universe is created out of nothing, but rather that the HH condition forces the condition that the universe bounce rather than enter or leave the potential singularity. Of course it is precisely at the bounce point that the role of the quantum corrections (here taken to be the imaginary part of the semi-classical action) have their largest value, and the interpretation of the universe following a classical path is most problematic.

We also note that our analysis demonstrates that , in the absence of non-homogeneous modes, the HH condition does not select the sign of the real part of the action (the parameter β above) and thus this analysis does not have anything to say to the controversy over this sign. This model however seems simple enough that it could also be used to address the issue of non-homogeneous fluctuations, at least to some low order of approximation, which could cast some light on the controversy. However, we believe that this would require a deeper understanding than we at least have of the role that such complex solutions really play in the evaluation of a path integral.

What has not been done here is to calculate the contribution of the "determinant", (the integration over non-classical fluctuations), to the wave function. It is possible that this would also lift the degeneracy over the various possible paths. This paper also does not address the broader question of the how one should use such complex solutions in evaluating the path integral. However, even in the context of ordinary quantum mechanics with a single degree of freedom, this question of how complex classical paths are to be used in general in evaluating the propagator is still one requiring more understanding.

ACKNOWLEDGEMENTS

WGU would like to thank NSERC and the Canadian Institute for Advanced Research for support while this work was carried out.

REFERENCES

- [1] J. Hartle, S.W. Hawking Phys Rev **D28** 2960(1983). Also S.W. Hawking in *AstrophysicalCosmology* ed H.A.Bruek, G.V. Coyne, M.S. Longair (Pontifica Academia Scientarium, Vatican City, 1982)
- [2] G. Gibbons, J. Hartle Phys Rev **D42** 2458 (1990)
- [3] J. Hartle, J.J. Halliwell Phys Rev **D41** 1815 (1990)
- [4] J.J. Halliwell, J. Louko Phys Rev **D42** 3997 (1990)
- [5] Glenn W. Lyons Phys Rev **D 46** 1546 (1992)
- [6] For an example of the dispute see Andrei Linde, gr-qc/9802038 “Quantum Creation of an Open Inflationary Universe” . See also the response to the criticism of in S.W. Hawking, Neil Turok gr-qc/9802062 “Comment on ‘Quantum Creation of an Open Universe’,by Andrei Linde”